PERTH MODERN SCHOOL

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INDEPENDENT PUBLIC SCHOOL

WAEP Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work:	five minutes
Working time:	fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet

Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

Let u, v and w represent complex numbers.

(a) If $u = 3 - 2i$ determine $u \times \bar{u}$. (2 marks)

Solution
$(3-2i)(3+2i) = 9 - 4i^2$ $= 13$
Specific behaviours
✓ indicates conjugate   ✓ expands and simplifies

(b) If $v = \dfrac{4+i}{2-i}$ determine $\mathrm{Re}(v)$. (2 marks)

Solution
$\dfrac{(4+i)(2+i)}{(2-i)(2+i)} = \dfrac{7+6i}{5}$ $\mathrm{Re}(v) = \dfrac{7}{5}$
Specific behaviours
✓ multiplies by conjugate   ✓ simplifies correctly and states real part

(c) If $w^2 - 6w + 10 = 0$, determine w. (2 marks)

Solution
$(w-3)^2 = -10 + 9$ $= -1$ $= i^2$ $w = 3 \pm i$
Specific behaviours
✓ completes square   ✓ both values of w

Question 2 **(6 marks)**

Consider the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$.

(a) Determine $\mathbf{b} - \mathbf{a}$. (1 mark)

Solution
$\mathbf{b} - \mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$
Specific behaviours
✓ correct difference

(b) Determine $|2\mathbf{a} + 3\mathbf{b}|$. (2 marks)

Solution
$2\mathbf{a} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}, 3\mathbf{b} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ $2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\lvert 2\mathbf{a} + 3\mathbf{b} \rvert = \sqrt{3^2 + (-1)^2} = \sqrt{10}$
Specific behaviours
✓ correct multiples ✓ correct sum and magnitude

(c) Given that $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{c}$, determine the value of the constant λ and the value of the constant μ. (3 marks)

Solution
Consider $\mathbf{i}$ coefficients: $3\lambda - \mu = 13$ Consider $\mathbf{j}$ coefficients: $-2\lambda + \mu = -10$ Add equations: $\lambda = 3$ Substitute: $-6 + \mu = -10 \Rightarrow \mu = -4$ $\lambda = 3, \mu = -4$
Specific behaviours
✓ writes two equations ✓ solves for λ ✓ solves for μ

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Question 3 **(8 marks)**

Let $A = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} b & 1 \\ 8 & b+2 \end{pmatrix}$, $C = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ and $D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where b is a constant.

(a) Simplify $AC + 3D$. (2 marks)

Solution
$AC + 3D = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 13 \\ -9 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ $= \begin{pmatrix} 19 \\ -6 \end{pmatrix}$
Specific behaviours
✓ correct product ✓ correctly adds product to multiple

(b) Determine the value(s) of b if B is singular. (3 marks)

Solution
$b(b+2) - 8 = 0$ $b^2 + 2b - 8 = 0$ $(b+2)(b-4) = 0$ $b = -2, b = 4$
Specific behaviours
✓ forms determinant and equates to 0 ✓ factorises quadratic ✓ correct values

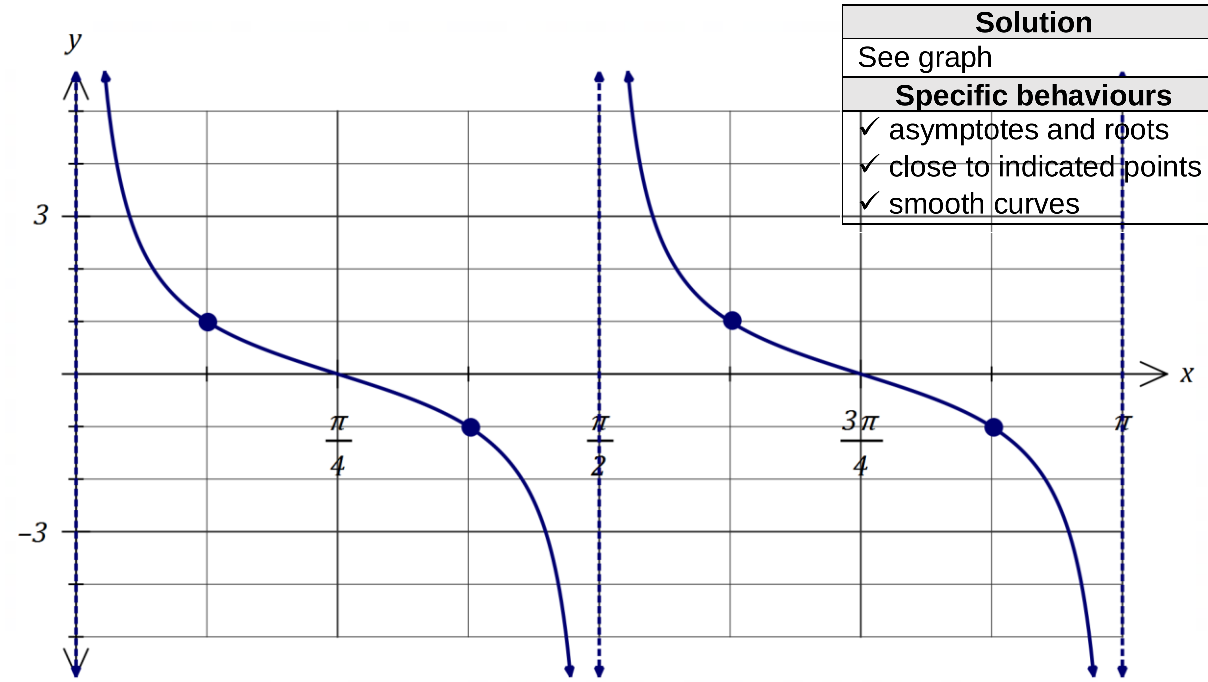
(c) Use a matrix method to determine X if $2X + AX = 4D$. (3 marks)

Solution
$(2I + A)X = 4D$ $X = (2I + A)^{-1} \times 4D$ $X = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^{-1} \times 4\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $= -\frac{1}{4}\begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix} \times 4\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} -2 \\ 5 \end{pmatrix}$
Specific behaviours
✓ expression for X ✓ correct inverse ✓ correct matrix X

Question 4 **(7 marks)**

(a) Sketch the graph of $y = \cot(2x)$ for $0 \leq x \leq \pi$. (3 marks)

Solution
See graph
Specific behaviours
✓ asymptotes and roots
✓ close to indicated points
✓ smooth curves



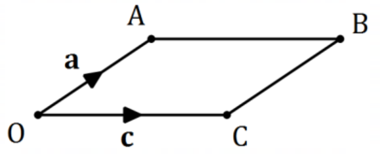
(b) Solve $\cos(2x) - \cos(x) = 0$ for $0 \leq x \leq 2\pi$. (4 marks)

Solution
$2\cos^2 x - 1 - \cos x = 0$ $(2\cos x + 1)(\cos x - 1) = 0$ $\cos x = 1 \Rightarrow x = 0, 2\pi$ $\cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$ $x = 0, x = \frac{2\pi}{3}, x = \frac{4\pi}{3}\ x = 2\pi$
Specific behaviours
✓ uses double angle identity   ✓ factorises   ✓ at least two correct solutions   ✓ all correct solutions

Question 5 **(5 marks)**

Let $OABC$ be a parallelogram and let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Sketch a diagram of $OABC$ and use a vector method to prove that the diagonals OB and AC intersect at right angles if and only if the parallelogram is a rhombus.

Solution



$$\overrightarrow{OB} = \mathbf{c} + \mathbf{a}$$
$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$$

For OB and AC to intersect at right angles then $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$.

$$\begin{aligned}\overrightarrow{OB} \cdot \overrightarrow{AC} &= (\mathbf{c} + \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) \\ &= \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} \\ &= \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{c}|^2 - |\mathbf{a}|^2\end{aligned}$$

Hence

$$\begin{aligned}|\mathbf{c}|^2 - |\mathbf{a}|^2 &= 0 \\ |\mathbf{c}|^2 &= |\mathbf{a}|^2 \\ |\mathbf{c}| &= |\mathbf{a}|\end{aligned}$$

Hence the diagonals of a parallelogram will only intersect at right angles if all side lengths are equal - the parallelogram is a rhombus.

Specific behaviours

✓ labelled diagram
✓ vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$
✓ forms and simplifies dot product
✓ shows that side lengths must be equal
✓ explains result

Question 6 **(6 marks)**

(a) Prove that $\sin(2x + x) = 3\sin x - 4\sin^3 x$. (3 marks)

Solution
$LHS = \sin(2x+x)$ $= \sin 2x \cos x + \cos 2x \sin x$ $= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x$ $= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$ $= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$ $= 3\sin x - 4\sin^3 x$ $= RHS$
Specific behaviours
✓ uses angle sum identity ✓ uses double angle identities ✓ uses Pythagorean identity and simplifies

(b) Solve $8\sin^3 x - 6\sin x + 1 = 0$ for $0 \le x \le \frac{\pi}{2}$. (3 marks)

Solution
$3\sin x - 4\sin^3 x = \frac{1}{2}$ $\sin 3x = \frac{1}{2}$ $3x = \frac{\pi}{6}, \frac{5\pi}{6}$ $x = \frac{\pi}{18}, \frac{5\pi}{18}$
Specific behaviours
✓ simplifies equation ✓ one correct value for $3x$ ✓ both solutions

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Question 7 **(7 marks)**

(a) If $\tan\theta = \frac{1}{2}$, determine the value of $\tan 2\theta$. (1 mark)

Solution
$\tan 2\theta = \dfrac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = 1 \div \frac{3}{4} = \frac{4}{3}$
Specific behaviours
✓ correct value

(b) Determine the transformation matrix for a reflection in the line $y = \frac{x}{2}$. (3 marks)

Solution
$\tan\theta = \frac{1}{2} \Rightarrow \tan 2\theta = \frac{4}{3}$ Using $3-4-5$ triangle, $\cos 2\theta = \frac{3}{5}$ and $\sin 2\theta = \frac{4}{5}$ $M = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$
Specific behaviours
✓ uses $m = \tan\theta$ ✓ values for $\cos 2\theta$, $\sin 2\theta$ ✓ correct matrix

(c) When reflected in $y = \frac{x}{2}$, the image of the point $(a, -5)$ is $(2, b)$. Determine the value of a and the value of b. (3 marks)

Solution
$\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} a \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ b \end{bmatrix}$ $3a - 20 = 10 \Rightarrow a = 10$ $b = \frac{1}{5}(4(10) + 15) = 11$ $a = 10, b = 11$
Specific behaviours
✓ matrix equation ✓ solves for a ✓ determines b

Question 8 **(7 marks)**

(a) Use a counter example to disprove that $n^2 + n + 11$ is prime for $n \in \mathbb{N}$. (2 marks)

Solution
When $n = 11$, then $n^2 + n + 11 = 11^2 + 11 + 11$ $= 11(11 + 1 + 1)$ $= 11(13)$ Hence $n^2 + n + 11$ has two factors and cannot be prime.
Specific behaviours
✓ suitable value of n ✓ valid justification

(b) Prove by induction that $7^n - 1$ is always divisible by 6 for $n \in \mathbb{N}$. (5 marks)

Solution
Let $f(n) = 7^n - 1$ and $P(n)$ be the statement that $f(n)$ is divisible by 6. Since $f(1) = 7^1 - 1 = 6$ then $P(1)$ is true. Assume that $P(k)$ is true and so $7^k - 1 = 6I, I \in N$. Then $f(k+1) = 7^{k+1} - 1$ $= 7.7^k - 1$ $= 6.7^k + 7^k - 1$ $= 6.7^k + 6I$ $= 6\left(7^k + I\right)$ Hence $P(k+1)$ is true if $P(k)$ is true. Since $P(1)$ is true then by the principle of induction it follows that $P(n)$ is always true.
Specific behaviours
✓ truth of initial case ✓ states assumption and uses to create $6I$ ✓ expression for $P(k+1)$ ✓ factors out 6 ✓ statement justifying truth

End of questions

Supplementary page

Question number: _________